1. An algebra teacher, testing her students on geometric sequences, asked her students to find the sum of all three terms of a geometric sequence in which the middle number was missing. Unfortunately, one student confused geometric sequences with arithmetic sequences, and then completed the problem with no other errors, obtaining an answer of 351. If all numbers involved were distinct positive integers, compute, with proof, the correct answer to the teacher's question.

2. The numbers $x_1, x_2, x_3, \ldots x_n$ are written on a chalkboard. -2(kboa)-6(r)3(d.0+941o8d1-oa)-4haooo8c

SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II   2009-10

1. Represent the terms of the geometric series with a, ar, ar^2. Then the student's sequence becomes $a, \frac{1}{2}(a + ar^2), ar^2$. Thus, $a + \frac{1}{2}(a + ar^2) + ar^2 = 351$.

   Simplifying, $3a + 3ar^2 = 702$. Solving this last equation for a, $a = \dfrac{234}{1+r^2}$.

   Noting that $234 = (2)(3^2)(13)$, this equation will only yield integral values of a for r = 1 and r = 5. If r = 1, the numbers in the teacher's sequence are not distinct. Therefore, r = 5 and the three numbers are 9, 45, and 225 with a sum of 279.

2. Proof by mathematical induction on the number of numbers, n.
   If n = 1, we certainly have $(1 + x_1) - 1 = x_1$

   Suppose the statement is true for all k < n numbers. Suppose the process has been repeated until only two numbers, a and b, are left, and suppose a was obtained by combining $x_1, x_2, \ldots x_m$ and b was obtained by combining $x_{m+1}, x_{m+2}, \ldots x_k$. Then by the induction hypothesis,

   $$a = (1+x_1)(1+x_2)\ldots(1+x_m) - 1 \text{ and } b = (1+x_{m+1})(1+x_{m+2})\ldots(1+x_k) - 1.$$

   Therefore, the final number is given by a + b + ab =

$$a + b + (1+x_1)(1+x_2)\ldots(1+x_k) - \underbrace{(1+x_1)(1+x_2)\ldots(1+x_m)}_{a+1} - \underbrace{(1+x_{m+1})(1+x_{m+2})\ldots(1+x_k)}_{b+1} + 1 =$$

$$a + b + (1+x_1)(1+x_2)\ldots(1+x_k) - (a+1) - (b+1) + 1 =$$

$$(1+x_1)(1+x_2)(1+x_3)\ldots(1+x_k) - 1$$

3. $\log_{\sin x}(\tan x) = \log_{\tan x}(\sin x) = \dfrac{1}{\log_{\sin x}(\tan x)}$ (by the change of base formula),

   where sin x > 0 and tan x > 0. Therefore, $(\log_{\sin x}(\tan x))^2 = 1$,

   which implies $\log_{\sin x}(\tan x) = \pm 1$.

   $\log_{\sin x}(\tan x) = 1 \longrightarrow \tan x = \sin x \longrightarrow \cos x = 1.$

   However, this would make sin x = 0.

   $\log_{\sin x}(\tan x) = -1 \longrightarrow \tan x = \dfrac{1}{\sin x} \longrightarrow \cos x = \sin^2 x = 1 - \cos^2 x$

   Therefore, $\cos^2 x + \cos x - 1 = 0 \longrightarrow \cos x = \dfrac{-1 \pm \sqrt{5}}{2}$.

   If cos x is negative, then one of tan x or sin x is negative also. Therefore, the only possible value of cos x is $\dfrac{-1 + \sqrt{5}}{2}$.

4. Let m‘ P = x and m‘ PAB = m‘ PAD = y.

Since PBC is an exterior angle of triangle PAB,
m‘ PBC= x+y and m‘ DBP= x+y also.
Therefore, m‘ DBC= 2x+2y. Since ‘ DBC is an exterior angle of triangle ABD,

m‘ DBC = m‘ D + m‘ DAB or

2x + 2y = m‘ D + 2y.

Therefore, m‘ D = 2x and the desired ratio is 2:1.

5. Let the area of the first T be equal to 1.

Denote by Aj the j-th patch and by Pj = |Aj| the area of j-th patch.
) X U W K H U Z H G H Q R W H E \ 3 L M _ $ L @ $ M _ W K H D U H D R I L tion of the i-th patch and j-th patch.
Analogously we define Pijk, Pijkl, and P