Solutions

1. Method 1: Represent the four numbers in arithmetic sequence as a, $a + d$, $a + 2d$, and $a + 3d$. Then, the geometric sequence is a, $a + d$ 3, $a + 2d$, and $a + 3d + 12$. Therefore,

$$ \underline{\quad\quad} = \underline{\quad\quad} \qquad (1)$$

Similarly, $\underline{\quad\quad} = \underline{\quad\quad}$.

Therefore, $a = d + 9$.
Substituting this last equation into (1) and simplifying,

$d = 15$, $d =$ 3.

If $d = 15$, $a = 24$, and the arithmetic sequence is 24, 39, 54, 69
If $d =$ 3, $a = 6$, and the arithmetic sequence is 6, 3, 0, 3.

A quick check shows that 24, 39, 54, 69 satisfies the conditions of the problem, with the corresponding geometric sequence being 24, 36, 54, 81.
However, 6, 3, 0, 3 does not work since 6, 0, 0, 9 is not a geometric sequence.
Therefore, the only arithmetic sequence is 24, 39, 54, 69.

Method 2: Represent the four numbers in arithmetic sequence as a, $a + d$, $a + 2d$, and $a + 3d$. Let the terms of the geometric sequence be represented by a, ar, , . Then

(1) $ar = a + d$ 3 and (2) $= a + 2d$ and (3) $a + 3d =$

From (1) $a(r$ 1$) = d$ 3. From (2) $a($ $) = 2d$ $a(r$ 1$)(r + 1) = 2d$.

Therefore, $(d$ 3$)(r + 1) = 2d$ (4) $r + 1 = \underline{\quad}$.

From (3) $= 3d + 12$ $a($ $) = 3d + 12$ $a(r$ 1$)($ $) = 3(d + 4)$

Substituting (1) into this last equation and dividing by d 3, we obtain

(5) ( ) = $\underline{\quad\quad}$.

Substituting (4) into (5) we obtain + $\underline{\quad}$ = $\underline{\quad\quad}$ = $\underline{\quad\quad}$ $\underline{\quad}$ = $\underline{\quad\quad}$.

From (4) $r = \underline{\quad}$ 1 = $\underline{\quad}$ = $\underline{\quad\quad}$.

Therefore, $\underline{\quad} = \underline{\quad\quad}$ from which we eventually obtain $d = 15$. Thus, from (4),

$r = \underline{\ }$ and from (1) $a = 24$. Therefore, the only such arithmetic sequence is 24, 39, 54, 69.

2. Assume that $f(x) = 0$ has an integer root . Since the lead coefficient of is 1, the sum of the roots is Since a is an integer, $f(x)$ has another integer root . Thus, $f(x) = (x\quad)(x\quad)$, and $f(300) = (300\quad)(300\quad)$. Without loss of generality, let .

Since we are given $f(300)$ is prime, this means that (300) = 1 and (300) is prime. Therefore, 299 while 7 (since 293 and 307 are the closest primes to 300). Since the product of the roots of $f(x)$ = is b, = 7·299 = 2093. But this is a contradiction, since we are given 2093.

Therefore, $f(x) = 0$ has no integer solutions.

3. Method 1: Construct diagonal . Since ADC ABC (SSS), ADC ABC. Therefore, m DCA = m BCA = ½ m ABC. Let m BCA = x and m ABC = $2x$, and let AC = a. Using the Law of Sines on ABC,

cos x = .

Using the Law of Cosines on ABC,

= + − a = 6. Then cos x = = , and cos2x = − 1 = .

Now, construct the altitude of PBC to , meeting at point M. Since PBC is isosceles (C B), M is the midpoint of . Thus, BM = MC = 2.5.

Then, in right PMB, cos B = = , and PB = 20.

Finally, using the Pythagorean Theorem

on PMB,

and PM = , or ,

which is the desired distance.

Method 2: Construct diagonal . Since ADB and CDB are both isosceles triangles, ADC ABC and both are congruent to BCD. Thus, PBC is isosceles. Let PB = x, PA = x 4, and PD = x 5.

Using the Law of Cosines on PAD,

(1) 16 =

Using the Law of Cosines on PBC,

25 = cos P = —.

Substituting into (1) above,

— .

Carefully simplifying this last equation, we obtain

Factoring, from which x = − (impossible) and x = 20.

Finally, construct the altitude of PM of PBC and noting that M is the midpoint of BC, use the Pythagorean Theorem on PMB.

and PM = —, or —,

which is the desired distance.

4. Assume that for some positive integer *a*.
We first prove that *n* is not a multiple of *p*.
Suppose that for some integer *k*. Then and, therefore,

Hence, *p* must divide *a* which means − is an integer, and − .

Then, $k <$ − $< k + 1$, which is impossible. Therefore, *n* is not a multiple of *p*.

Next, we prove that *n* and $n + p$ have no common prime factors. Suppose a prime divides both *n* and $n +$

5. The desired ratio is – .

Method 1

Construct and . Represent the area of ABC as [ABC].

[AFE] = [AFD], since DF = FE, and AFE and AFD have the same altitude from point A. Similarly, [BFE] = [BFD].

Thus, [AEB] = 2[AFB],

[AFE] = 2[EFC], since AE = 2(EC) and AFE and EFC have the same altitude from point F. Similarly, [BFD] = 2[AFD] = 2[AFE] = 4[EFC].

Also, [ADE] = [AFD] + [AFE] = 4[EFC].

[AEB] = – [ABC], since AE = – AC and the triangles have the same altitude from point B.

Therefore, [AEB] = 2[AFB] = – [ABC]   [AFB] = – [ABC]

Also, [AFB] = [AFD] + [BFD] = [AFE] + [BFD]
= 2[EFC] + 2[AFD] = 2[EFC] + 4[EFC] = 6[EFC] .

Therefore, [AFB] = – [ABC] = 6[EFC]   [EFC] = — [ABC].

Finally, [BFC] = [ABC] [EFC] [ADE] [BFD]

= [ABC] — [ABC] — [ABC] — [ABC] = –[ABC].

Method 2

Construct perpendiculars from D, A, F, and E to , and label the points of intersection , , , and , respectively.

The area of ABC = –

Since D is parallel to A , D B is similar to A B.

Therefore, —— ——

Method 3

Let EC = x, EA = $2x$, AD = y, BD = $2y$, and DF = EF = w.

Let ADE = and AED = .

Area ABC = – = ———.

Area AED = – = .

Area EFC = – – .

Area FDB = – .

Using the Law of Sines on AED,